17 BEAMS SUBJECTED TO TORSION AND BENDING -I

1.0 INTRODUCTION

When a beam is transversely loaded in such a manner that the resultant force passes through the longitudinal shear centre axis, the beam only bends and no torsion will occur. When the resultant acts away from the shear centre axis, then the beam will not only bend but also twist.

When a beam is subjected to a pure bending moment, originally plane transverse sections before the load was applied, remain plane after the member is loaded. Even in the presence of shear, the modification of stress distribution in most practical cases is very small so that the Engineer's Theory of Bending is sufficiently accurate.

If a beam is subjected to a twisting moment, the assumption of planarity is simply incorrect except for solid circular sections and for hollow circular sections with constant thickness. Any other section will warp when twisted. Computation of stress distribution based on the assumption of planarity will give misleading results. Torsional stiffness is also seriously affected by this warping. If originally plane sections remained plane after twist, the torsional rigidity could be calculated simply as the product of the polar moment of inertia ($I_p = I_{xx} + I_{yy}$) multiplied by (G), the shear modulus, viz. $G. (I_{xx} + I_{yy})$. Here I_{xx} and I_{yy} are the moments of inertia about the principal axes. This result is accurate for the circular sections referred above. For all other cases, this is an overestimate; in many structural sections of quite normal proportions, the true value of torsional stiffness as determined by experiments is only *1% - 2%* of the value calculated from polar moment of inertia.

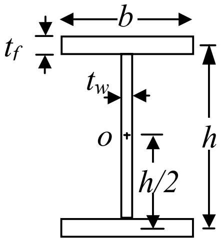
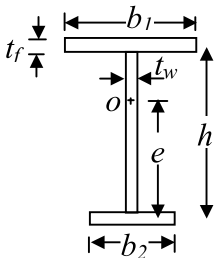
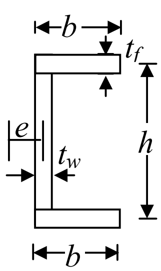
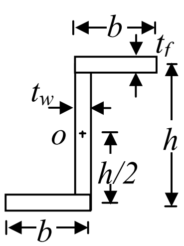
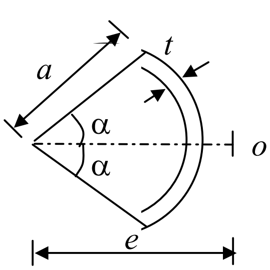
It should be emphasised that the end sections of a member subjected to warping may be modified by constraints. If the central section remains plane, for example, due to symmetry of design and loading, the stresses at this section will differ from those based on free warping. Extreme caution is warranted in analysing sections subjected to torsion.

2.0 UNIFORM AND NON-UNIFORM TORSION

2.1 Shear Centre and Warping

Shear Centre is defined as the point in the cross-section through which the lateral (or transverse) loads must pass to produce bending without twisting. It is also the centre of rotation, when only pure torque is applied. ***The shear centre and the centroid of the cross section will coincide, when section has two axes of symmetry. The shear centre will be on the axis of symmetry, when the cross section has one axis of symmetry.***

Table 1: Properties of Sections

	$J = \dfrac{2bt_f^3 + ht_w^3}{3}$ $C_w = \dfrac{t_f h^2 b^3}{24}$	*If* $t_f = t_w = t$: $J = \dfrac{t^3}{3}(2b+h)$
	$e = h\,\dfrac{b_1^3}{b_1^3 + b_2^3}$ $J = \dfrac{(b_1+b_2)t_f^3 + ht_w^3}{3}$ $C_w = \dfrac{t_f h^2}{12}\,\dfrac{b_1^3 b_2^3}{b_1^3 + b_2^3}$	*If* $t_f = t_w = t$: $J = \dfrac{t^3}{3}(b_1 + b_2 + h)$
	$e = \dfrac{3b^2 t_f}{6bt_f + ht_w}$ $J = \dfrac{2bt_w^3 + ht_w^3}{3}$ $C_w = \dfrac{t_f b^3 h^2}{12}\,\dfrac{3bt_f + 2ht_w}{6bt_f + ht_w}$	*If* $t_f = t_w = t$: $e = \dfrac{3b^2 t_f}{6b+h}$ $J = \dfrac{t^3}{3}(2b+h)$ $C_w = \dfrac{tb^3h^2}{12}\,\dfrac{3b+2h}{6b+h}$
	$J = \dfrac{2bt_f^3 + ht_w^3}{3}$ $C_w = \dfrac{b^3h^2}{12(2b+h)^2}$ $\times[2t_f(b^2 + bh + h^2) + 3t_w bh]$	*If* $t_f = t_w = t$: $J = \dfrac{t^3}{3}(2b+h)$ $C_w = \dfrac{tb^3h^2}{12}\,\dfrac{b+2h}{2b+h}$
	$e = 2a\,\dfrac{\sin\alpha - \alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha}$ $J = \dfrac{2a\alpha t^3}{3}$ $C_w = \dfrac{2ta^5}{3}$ $\times\left[\alpha^3 - \dfrac{6(\sin\alpha - \alpha\cos\alpha)^2}{\alpha - \sin\alpha\cos\alpha}\right]$	*If* $2\alpha = \pi$: $e = \dfrac{4a}{\pi}$ $J = \dfrac{\pi a t^3}{3}$ $C_w = \dfrac{2ta^5}{3}\left(\dfrac{\pi^3}{8} - \dfrac{12}{\pi}\right) = 0.0374ta^5$

where O = shear centre; J = torsion constant; C_w = warping constant

If the loads are applied away from the shear centre axis, torsion besides flexure will be the evident result. The beam will be subjected to stresses due to torsion, as well as due to bending.

The effect of torsional loading can be further split into two parts, the first part causing twist and the second, *warping.* These are discussed in detail in the next section.

Warping of the section does not allow a plane section to remain as plane after twisting. This phenomenon is predominant in Thin Walled Sections, although consideration will have to be given to warping occasionally in hot rolled sections. An added characteristic associated with torsion of non-circular sections is the in-plane distortion of the cross-section, which can usually be prevented by the provision of a stiff diaphragm. Distortion as a phenomenon is not covered herein, as it is beyond the scope of this chapter.

Methods of calculating the position of the shear centre of a cross section are found in standard textbooks on Strength of Materials.

2.2 Classification of Torsion as Uniform and Non-uniform

As explained above when torsion is applied to a structural member, its cross section may warp in addition to twisting. If the member is allowed to warp freely, then the applied torque is resisted entirely by torsional shear stresses (called ***St. Venant's torsional shear stress***). If the member is not allowed to warp freely, the applied torque is resisted by St. Venant's torsional shear stress and warping torsion. This behaviour is called ***non-uniform torsion.***

Hence (as stated above), the effect of torsion can be further split into two parts:

- Uniform or Pure Torsion (called St. Venant's torsion) - T_{sv}
- Non-Uniform Torsion, consisting of St.Venant's torsion (T_{sv}) and warping torsion (T_w).

2.3 Uniform Torsion in a Circular Cross Section

Let us consider a bar of constant circular cross section subjected to torsion as shown in Fig. 1. In this case, *plane cross sections normal to the axis of the member remain plane after twisting*, i.e. there is no warping. The torque is solely resisted by circumferential shear stresses caused by St. Venant's torsion. Its magnitude varies as its distance from the centroid.

For a circular section, the St. Venant's torsion is given by

$$T_{sv} = I_p G \frac{d\phi}{dz} \tag{1}$$

where, ϕ - angle of twist
G - modulus of rigidity
T_{sv} - St. Venant's torsion.
I_p - the polar moment of inertia
z - direction along axis of the member.

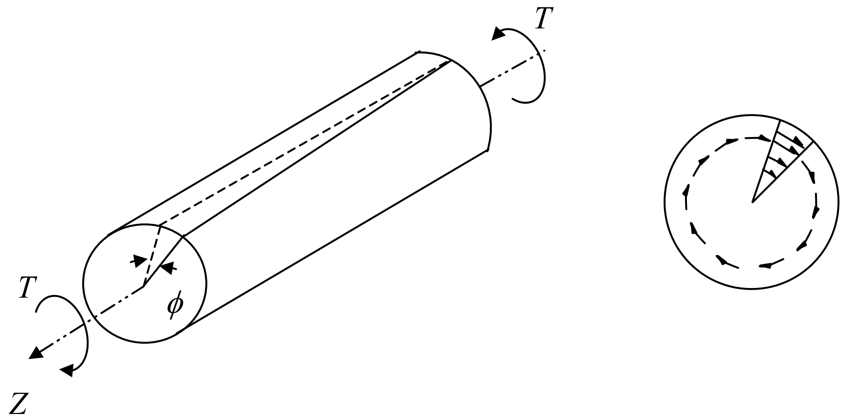


Fig. 1Twisting of circular section.

2.4 Uniform Torsion in Non-Circular Sections

When a torque is applied to a non-circular cross section (e.g. a rectangular cross section), the transverse sections which are plane prior to twisting, warp in the axial direction, as described previously, so that a plane cross section no longer remains plane after twisting. However, so long as the warping is allowed to take place freely, the applied load is still resisted by shearing stresses similar to those in the circular bar. The St.Venant's torsion (T_{sv}) can be computed by an equation similar to equation (1) but by replacing I_p by J, the torsional constant. The torsional constant *(J)* for the rectangular section can be approximated as given below:

$$J = C.\, bt^3 \qquad (1.a)$$

where b and t are the breadth and thickness of the rectangle. C is a constant depending upon (b/t) ratio and tends to *1/3* as b/t increases.

Then ,
$$T_{sv} = JG\frac{d\phi}{dz} \qquad (1.b)$$

2.4.1 Torsional Constant (J) for thin walled open sections made up of rectangular elements

Torsional Constant *(J)* for members made up of rectangular plates (see Fig. 2) may be computed approximately from

$$J = \frac{1}{3} \sum_i b_i (t_i)^3 \qquad (1.c)$$

in which b_i and t_i are length and thickness respectively of any element of the section.

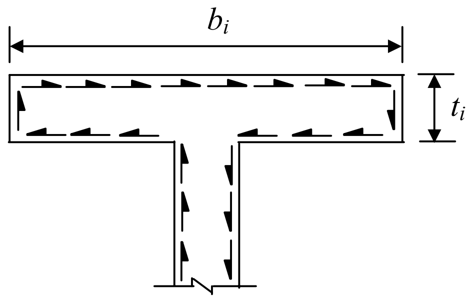


Fig. 2. Thin walled open section made of rectangular elements

In many cases, only uniform (or St. Venant's) torsion is applied to the section and the rate of change of angle of twist is constant along the member and the ends are free to warp (See Fig. 3)

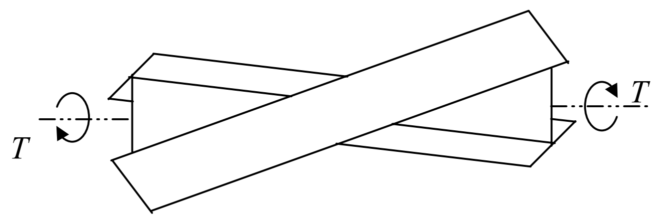


Fig.3 Uniform Torsion (Constant Torque : Ends are free to warp)

In this case the applied torque is resisted entirely by shear stresses and ***no warping stresses result.***

The total angle of twist ϕ is given by

$$\phi = \frac{T\ z}{GJ} \qquad (2)$$

where T = Applied Torsion = T_{sv}

(Note: in this case only St.Venant's Torsion is applied)

The maximum shear stress in the element of thickness t is given by

$$\tau_t = G t \phi' \qquad (3)$$

Fig. 4 gives the corresponding stress pattern for an I section.

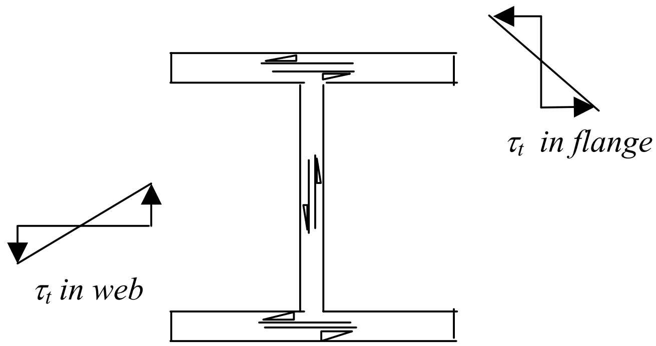


Fig.4 Stress pattern due to pure torsion
(Shear stresses are enlarged for clarity)

3.0 NON-UNIFORM TORSION

When warping deformation is constrained, the member undergoes non-uniform torsion. Non-uniform torsion is illustrated in Fig. 5 where an *I*-section fixed at one end is subjected to torsion at the other end. Here the member is restrained from warping freely as one end is fixed. The warping restraint causes bending deformation of the flanges in their plane in addition to twisting. The bending deformation is accompanied by a shear force in each flange.

The total non-uniform torsion (T_n) is given by

$$T_n = T_{sv} + T_w \tag{4}$$

where T_w is the warping torsion.

Shear force V_f in each flange is given by

$$V_f = -\frac{dM_f}{dz} \tag{5}$$

where M_f is the bending moment in each flange. Since, the flanges bend in opposite directions, the shear forces in the two flanges are oppositely directed and form a couple. This couple, which acts to resist the applied torque, is called ***warping torsion.***

For the *I*-section shown in Fig. 5, warping torsion is given by

$$T_w = V_f . h \tag{6}$$

The bending moment in the upper flange is given by

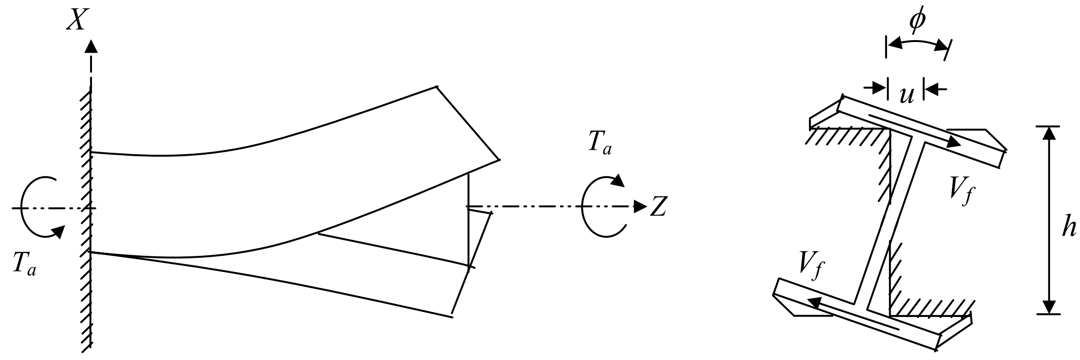


Fig. 5 Non uniform Torsion:Twisting of Non-Circular Section restrained against free warping (Constant Torque : End warping is prevented)

$$M_f = EI_f \frac{d^2u}{dz^2} \tag{7}$$

in which I_f is the moment of inertia of flange about its strong axis (i.e. the vertical axis) and u, the lateral displacement of the flange centreline which is given by

$$u = \phi h/2 \tag{8}$$

On substituting eq. 8 in eq. 7 we get

$$M_f = \frac{EI_f h}{2}\frac{d^2\phi}{dz^2} = \frac{EI_f\ h}{2}\ \phi'' \tag{9}$$

On simplification by substituting eqn.(9) into eqn. (6), we obtain the value of warping torsion as,

$$T_w = -\frac{EI_f h^2}{2}\frac{d^3\phi}{dz^3} = -\frac{EI_f h^2}{2}\phi''' \tag{10}$$

The term $I_f h^2/2$ is called the warping constant (Γ) for the cross-section.

then, $$T_w = -E\Gamma \frac{d^3\phi}{dz^3} = -E\Gamma\,\phi''' \tag{11}$$

in which $$\Gamma = \frac{I_f \cdot h^2}{2} \quad \text{(for an I-section)} \tag{12}$$

$E\Gamma$ is termed as the warping rigidity of the section, analogous to GJ, the St. Venant's torsional stiffness. The torque will be resisted by a combination of St.Venant's shearing stresses and warping torsion. Non-uniform torsional resistance (T_n) at any cross-section is therefore given by the sum of St.Venant's torsion (T_{sv}) and warping torsion (T_w).

Thus, the differential equation for non-uniform torsional resistance $T_n(z)$ can be written as the algebraic sum of the two effects, due to St.Venant's Torsion and Warping Torsion.

$$T_n(z) = GJ\frac{d\phi}{dz} - E\Gamma\frac{d^3\phi}{dz^3} = GJ\phi' - E\Gamma\,\phi''' \quad (13a)$$

$$or,\ T_n(z) = GJ\phi' - EI_f.\frac{h^2}{2}.\phi''' \quad \text{(for an } I\text{-section)} \quad (13b)$$

In the above, the first term on the right hand side (depending on GJ) represents the resistance of the section to twist and the second term represents the resistance to warping and is dependent on $E\Gamma$.

In the example considered (Fig. 5), the applied torque T_a is constant along the length, λ, of the beam . For equilibrium, the applied torque, T_a, should be equal to torsional resistance T_n.

The boundary conditions are: (i) the slope of the beam is zero when $z = 0$ and (ii) the BM is zero when $z = \lambda$ i.e. at the free end.

$$\frac{d\phi}{dz} = 0 \qquad when \qquad z = 0$$

$$\frac{d^2\phi}{dz^2} = 0 \qquad when \qquad z = \lambda$$

The solution of equation (13.a) is

$$\frac{d\phi}{dz} = \frac{T_n}{GJ}\left(1 - \frac{Cos\,h\dfrac{\lambda - z}{a}}{Cos\,h\dfrac{\lambda}{a}}\right) \quad (14)$$

in which $$a^2 = \frac{E\Gamma}{GJ} \quad (15)$$

Since the flexural rigidity EI_f and torsional rigidity GJ are both measured in the same units (N.mm ^{2}), equation (15) shows that a has the dimensions of length and *depends on the proportions of the beam*. Because of the presence of the second term in equation (14) the angle of twist per unit length varies along the length of the beam even though the applied torsion, T_a ,remains constant. When $\frac{d\phi}{dz}$ is known, the St. Venant's torsion (T_{sv}) and the warping torsion (T_w) may be calculated or any cross section. At the built-in section ($z = 0$) and $\frac{d\phi}{dz} = 0$, hence we obtain from eq.(1) that $T_{sv} = 0$. At this point, the entire torque is balanced by the moment of the shearing forces in each of the flanges.

$$\therefore V_f = -T_n/h \qquad (16)$$

At the end $z = \lambda$, using equation (14) , we obtain

$$\frac{d\phi}{dz} = \frac{T_n}{GJ}\left(1 - \frac{1}{Cos\,h\left(\lambda/a\right)}\right) \qquad (17)$$

If the length of the beam is large in comparison with the cross sectional dimensions, $\left(1 - \frac{1}{Cosh\left(\frac{\lambda}{a}\right)}\right)$ tends to approach 1, as the second term is negligible. Hence $\frac{d\phi}{dz}$

approaches T_n/GJ .

The bending moment in the flange is found from

$$V_f = \frac{dM_f}{dz} = EI_f \cdot \frac{d^3\phi}{dz^3} \cdot \frac{h}{2} \qquad (18)$$

where M_f is the bending moment in each flange.

$$M_f = EI_f \cdot \frac{h}{2} \cdot \frac{d^2\phi}{dz^2} \qquad (19)$$

Substituting for $\frac{d\phi}{dz}$ from eq. (14) we obtain

$$M_f = \frac{a}{h} . T_n \cdot \frac{Sin\,h\left(\frac{\lambda - z}{a}\right)}{Cos\,h\,\lambda/a} \qquad (20)$$

The maximum bending moment at the fixed end is given by

$$M_{f\,\max} = \frac{a}{h} \cdot T_n \cdot tanh\left(\frac{\lambda}{a}\right) \qquad (21)$$

When λ is several times larger than a, $tan\,h\,(\lambda/a)$ approaches 1, so that

$$M_{f\,\max} \cong \frac{a.T_n}{h} \qquad (22)$$

In other words, the maximum bending moment in each of the flanges will be the same as that of cantilever of length *a*, and loaded at the free end by a force of $\left(\frac{T_n}{h}\right)$. For a short beam λ is small in comparison with *a*, so $tanh\left(\frac{\lambda}{a}\right) \cong \left(\frac{\lambda}{a}\right)$

Hence $M_{f\,max} = \frac{T_n \cdot \lambda}{h}$ (23)

The range of values for $M_{f\,max}$ therefore varies from $\frac{T_n}{h}(\lambda)$ *to* $\frac{T_n}{h}(a)$ as the length of the beam varies from a "short" to a "long" one.

To calculate the angle of twist, ϕ, we integrate the right hand side of equation (14)

$$\phi = \frac{T_n}{GJ}\left[z + \frac{a \sin h\left(\frac{\lambda - z}{a}\right)}{Cos\, h\left(\frac{\lambda}{a}\right)} - a \tan h\left(\frac{\lambda}{a}\right)\right] \tag{24}$$

From equation (24), we obtain the value of ϕ at the end (i.e.) when $z = \lambda$

$$(\phi)_{z=\lambda} = \frac{T_n}{GJ}\left(\lambda - a \tan h\frac{\lambda}{a}\right) \tag{25}$$

For long beams $\tan h\left(\frac{\lambda}{a}\right) \cong 1$, so equation (25) becomes

$$(\phi)_{z=\lambda} = \frac{T_n}{GJ}(\lambda - a) \tag{26}$$

The effect of the warping restraint on the angle of twist is equivalent to diminishing the length λ of the beam to *(λ - a).*

Certain simple cases of the effect of Torsion in simply supported beams and cantilever are illustrated in Figures 6 and 7.

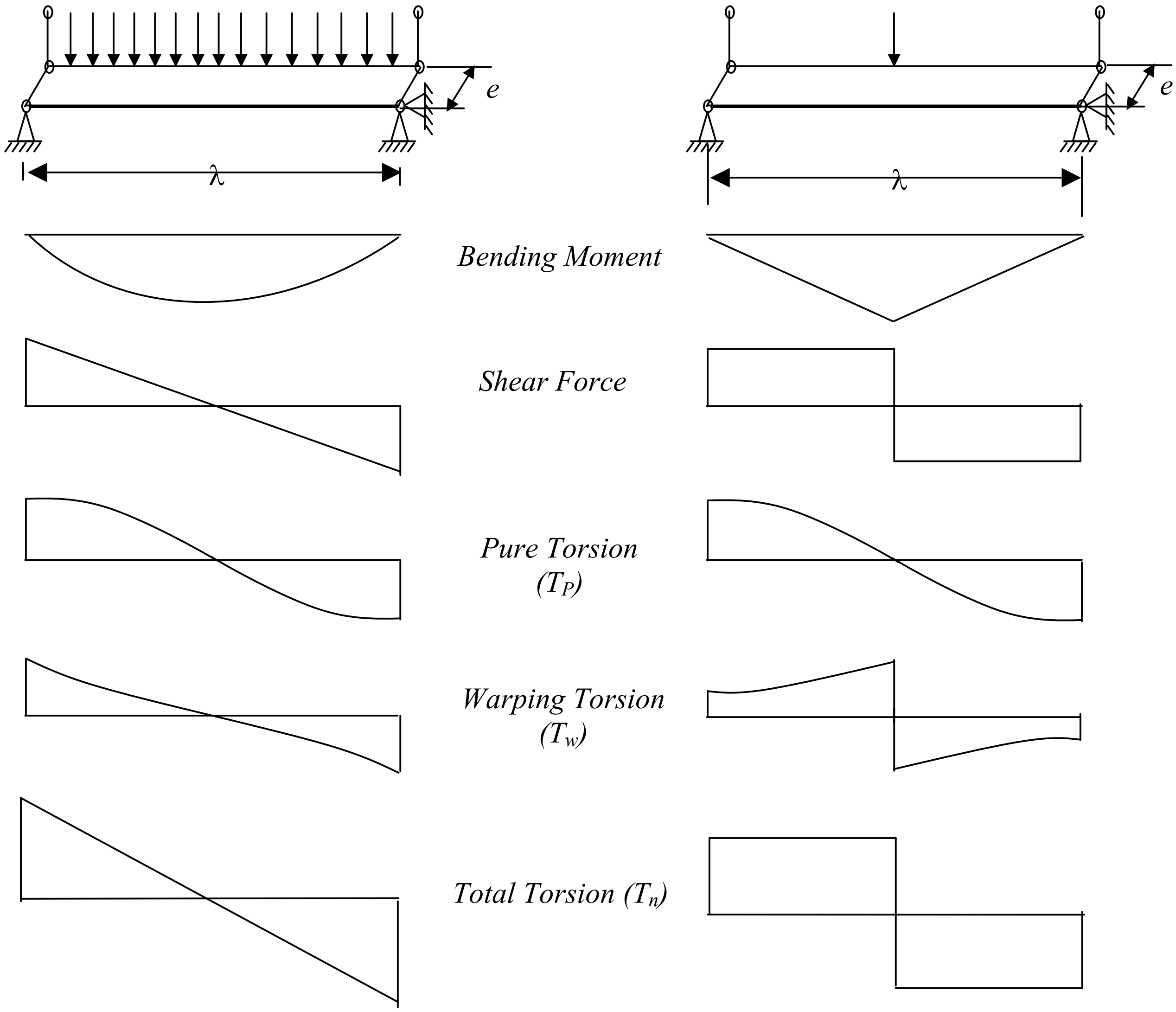


Fig.6 Torsion in simply supported beam with free end warping

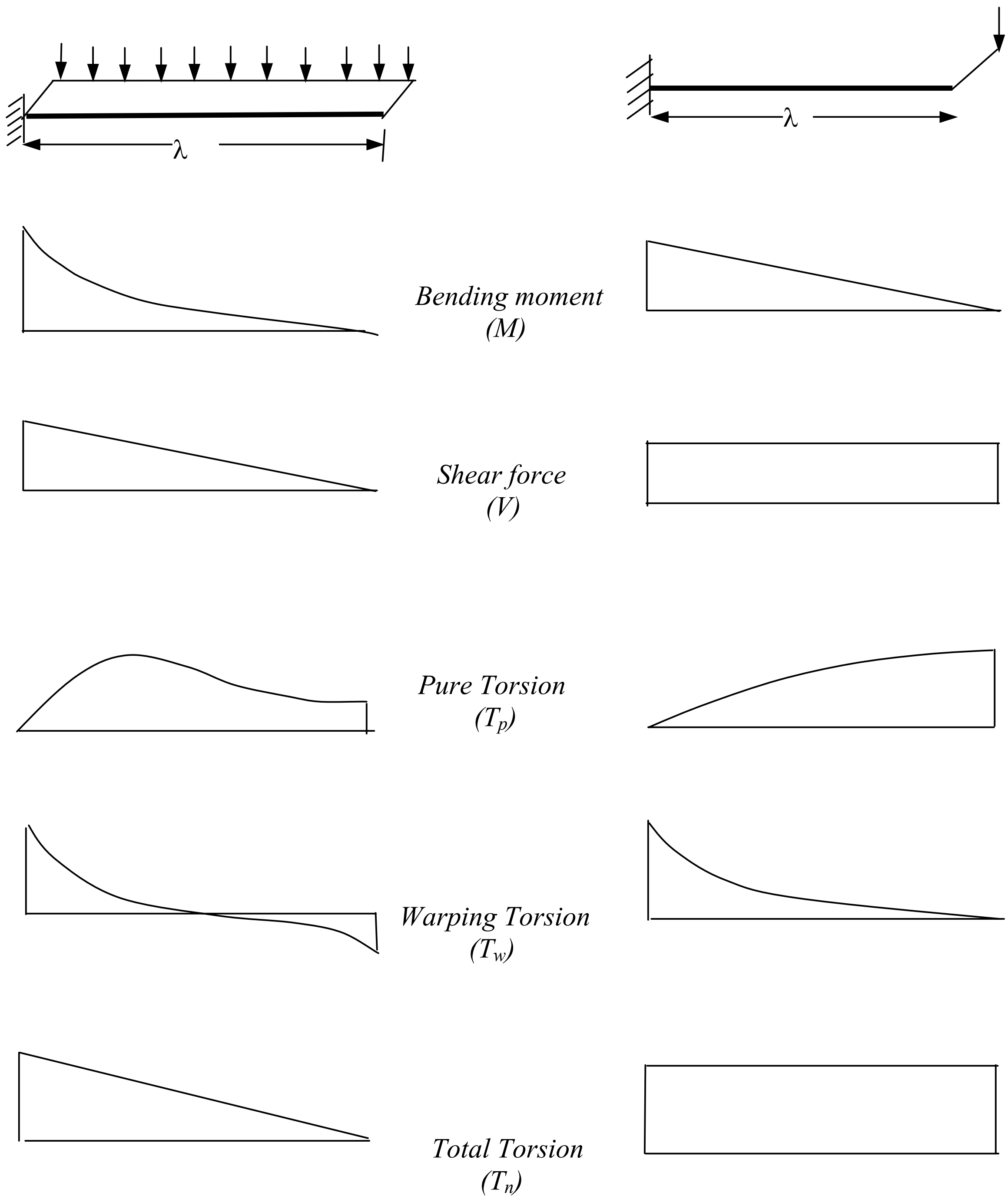


Fig.7 Torsion in Cantilevers

4.0 AN APPROXIMATE METHOD OF TORSION ANALYSIS

A simple approach is often adopted by structural designers for rapid design of steel structures subjected to torsion. This method (called ***the bi-moment method***) is sufficiently accurate for practical purposes. The applied torque is replaced by a couple of horizontal forces acting in the plane of the top and bottom flanges as shown in Fig. 8 and Fig. 9.

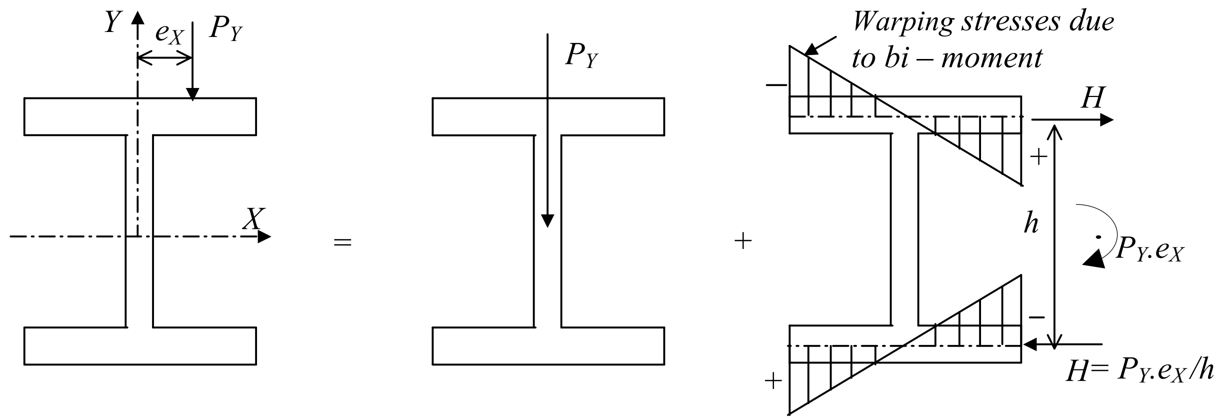


Fig 8: Load P_Y acting eccentrically w.r.t. y – axis and causing torsion.

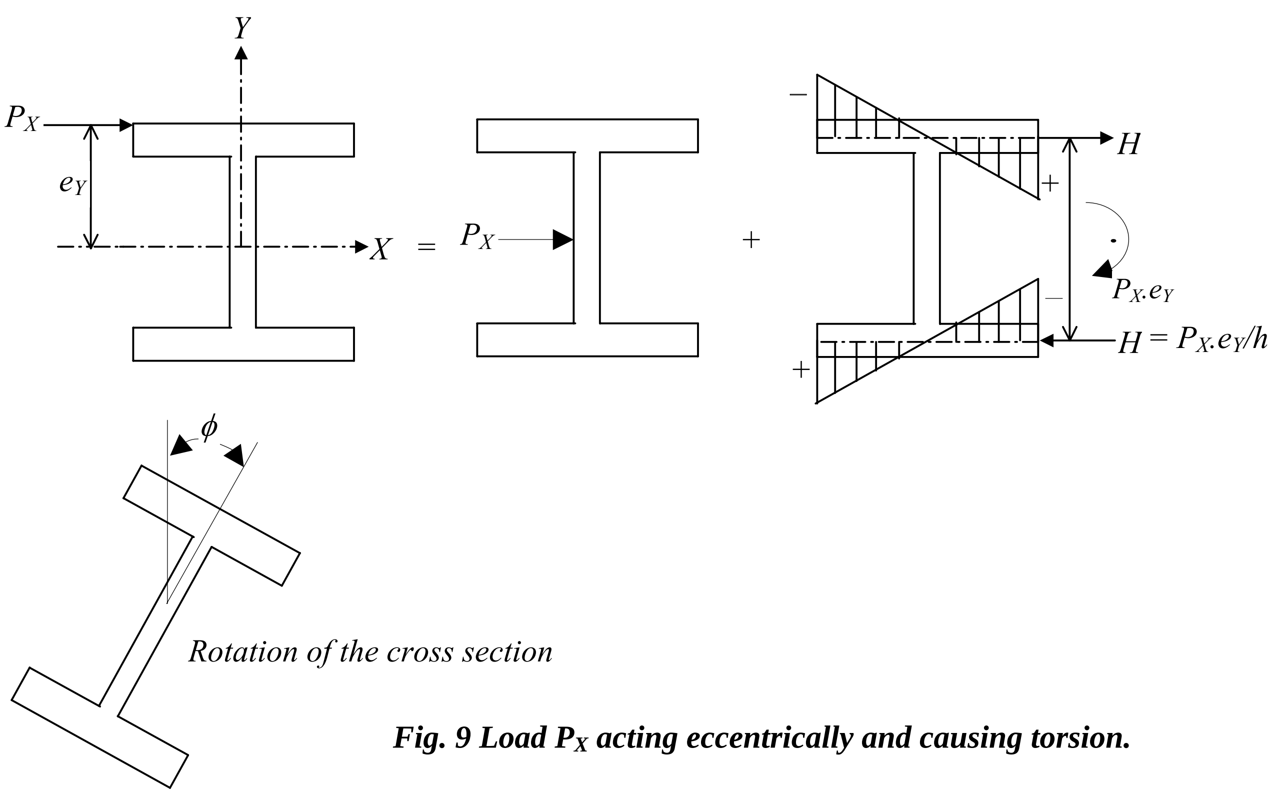


Fig. 9 Load P_X acting eccentrically and causing torsion.

When a uniform torque is applied to an open section restrained against warping, the member itself will be in non-uniform torsion. The angle of twist, therefore, varies along the member length. The rotation of the section will be accompanied by bending of flanges in their own plane. The direct and shear stresses caused are shown in Fig.10.

For an *I* section, the warping resistance can be interpreted in a simple way. The applied torque T_a is resisted by a couple comprising the two forces *H*, equal to the shear forces in each flange. These forces act at a distance equal to the depth between the centroids of each flange.

Each of these flanges can be visualized as a beam subjected to bending moments produced by the forces *H*. This leads to bending stresses σ_w in the flanges. These are termed ***Warping Normal Stresses.***

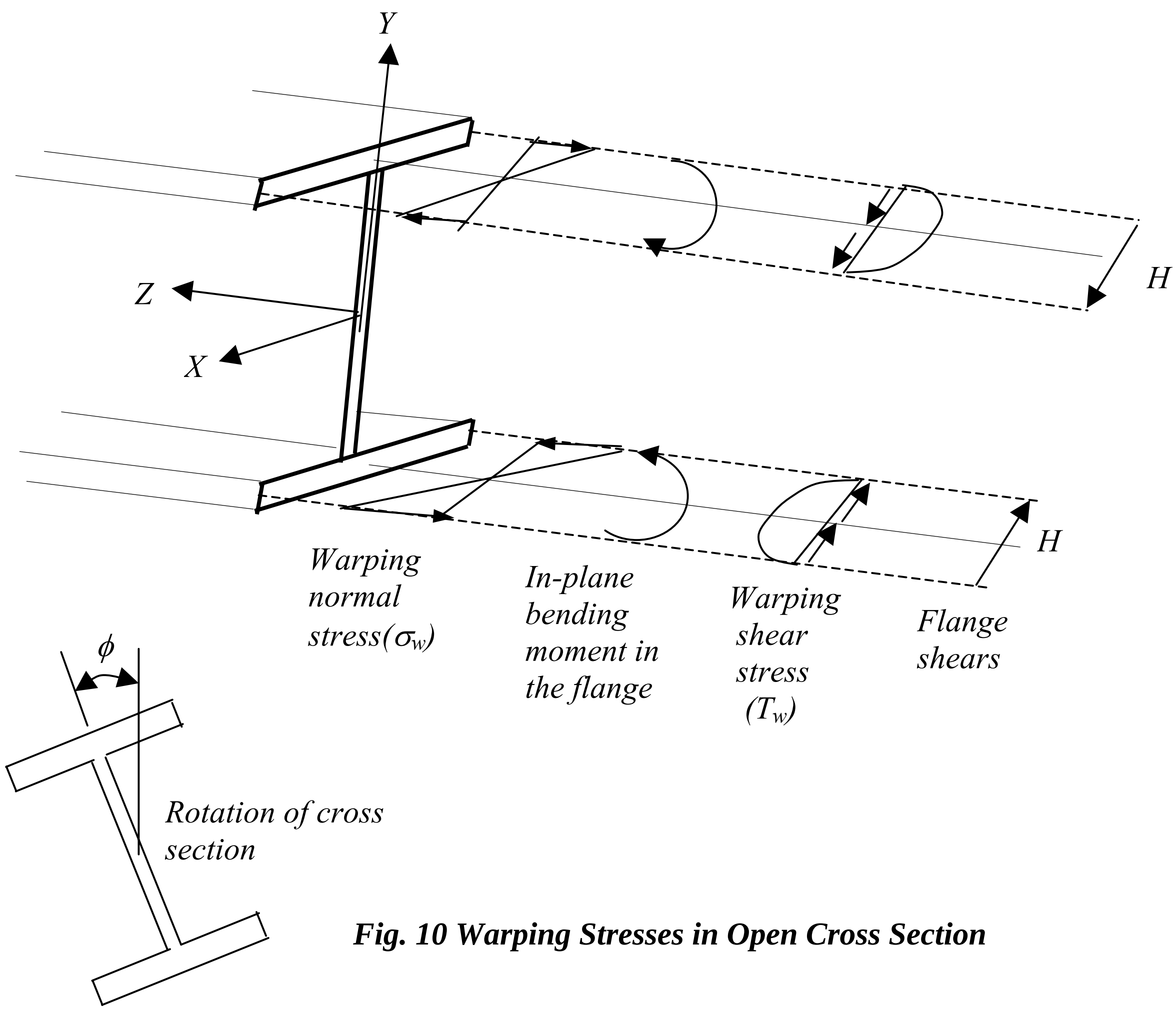


Fig. 10 Warping Stresses in Open Cross Section

The magnitude of the warping normal stress at any particular point *(σ_w)* in the cross section is given by

$$\sigma_w = -EW_{nwfs}\,\phi'' \tag{27}$$

where W_{nwfs} = normalised warping function at a particular point S in the cross section.

An approximate method of calculating the normalised warping function for any section is described in Reference 3. The value of W_{nwfs} for an I -section is given in section 5.3. The in-plane shear stresses are called ***Warping shear stresses***. They are constant across the thickness of the element. Their magnitude varies along the length of the element. The magnitude of the warping shear stress at any given point is given by

$$\tau_w = -\frac{ES_{wms}\,\phi'''}{t} \qquad (28)$$

where S_{wms} = Warping statical moment of area at a particular point S. Values of warping normal stress and in-plane shear stress are tabulated in standard steel tables produced by steel makers. Section 5.3 gives these values for I and H sections.

5.0 THE EFFECT OF TORSIONAL RIGIDITY *(GJ)* AND WARPING RIGIDITY *(EΓ)*

The warping deflections due to the displacement of the flanges vary along the length of the member. Both direct and shear stresses are generated in addition to those due to bending and pure torsion. As discussed previously, the stiffness of the member associated with the former stresses is directly proportional to the warping rigidity, $E\Gamma$.

When the torsional rigidity (GJ) is very large compared to the warping rigidity, $E\Gamma$, then the section will effectively be in "uniform torsion". Closed sections (eg. rectangular or square hollow sections) angles and Tees behave this way, as do most flat plates and all circular sections. Conversely if GJ is very small compared with $E\Gamma$, the member will effectively be subjected to warping torsion. Most thin walled open sections fall under this category. Hot rolled I and H sections as well as channel sections exhibit a torsional behaviour in between these two extremes. In other words, the members will be in a state of non-uniform torsion and the loading will be resisted by a combination of uniform (St.Venant's) and warping torsion.

5.1 End Conditions

The end support conditions of the member influence the torsional behaviour significantly; three ideal situations are described below. (It must be noted that torsional fixity is essential at least in one location to prevent the structural element twisting bodily). Warping fixity cannot be provided without also ensuring torsional fixity.

The following end conditions are relevant for torsion calculations

- ***Torsion fixed, Warping fixed***: This means that the twisting along the longitudinal (Z) axis and also the warping of cross section at the end of the member are prevented. ($\phi = \phi' = 0$ at the end). This is also called "fixed" end condition.

- ***Torsion fixed, Warping free:*** This means that the cross section at the end of the member cannot twist, but is allowed to warp. ($\phi = \phi'' = 0$). This is also called "pinned" end condition.

- ***Torsion free, Warping free***: This means that the end is free to twist and warp. The unsupported end of cantilever illustrates this condition. (This is also called "free" end condition).

Effective warping fixity is difficult to provide. It is not enough to provide a connection which provides fixity for bending about both axes. It is also necessary to restrain the flanges by additional suitable reinforcements. It may be more practical to assume "warping free" condition even when the structural element is treated as "fixed" for bending. On the other hand, torsional fixity can be provided relatively simply by standard end connections.

5.2 Procedures for checking adequacy in Flexure

These procedures have been described in an earlier chapter dealing with "unrestrained bending". Particular attention should be paid to lateral torsional buckling by evaluating the equivalent uniform moment $\overline{M}$, such that

$$\overline{M} < M_b$$

where $\overline{M}$ = equivalent uniform moment
M_b = lateral-torsional buckling resistance moment.

If the beam is stocky (eg. due to closely spaced lateral restraints), the design will be covered by moment capacity M_c.

In addition to bending stresses the shear stresses, τ_b, due to plane bending have to be evaluated.

Shear stress at any section is given by, $\tau = \dfrac{V A \bar{y}}{I t} = \dfrac{V Q}{I t}$

where Q = Statical moment of area of the shaded part (Fig. 11).

For the web, $\tau_{bw} = \dfrac{VQ_w}{I.t}$

For the flange, $\tau_{bf} = \dfrac{VQ_f}{I.T}$

where V = applied shear force
I = moment of inertia of the whole section

T = flange thickness
Q_w = statical moment of area for the web
Q_f = statical moment of area for the flange.
t = web thickness

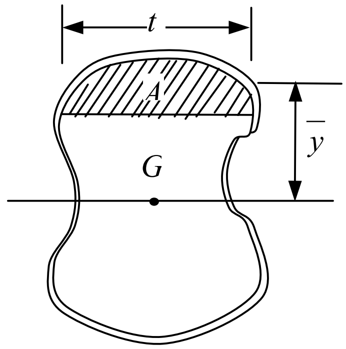


Fig.11

5.3 Cross Sectional Properties for Symmetrical *I* and *H* Sections

For an I or H section subjected to torsion, the following properties will be useful (see Fig. 12).

$$J = \frac{1}{3}\left[2BT^3 + (D-2T)t^3\right]$$

$$W_{nwfs} = \frac{hB}{4}$$

$$S_{wms} = \frac{hB^2T}{16}$$

$$\Gamma = \frac{I_y h^2}{4}$$

$$Q_f = A_f \cdot y_f$$

$$Q_w = \frac{A}{2} y_w$$

where A_f = area of half the flange
y_f = distance of neutral axis to the centroid of the area A_f
A = total cross sectional area
y_w = the distance from the neutral axis to the centroid of the area above neutral axis.

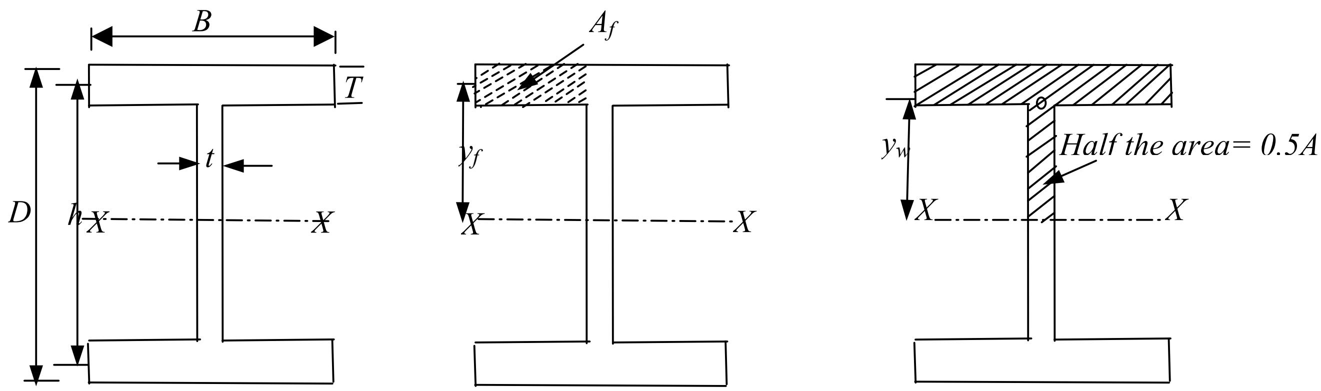


Fig.12

6.0 CONCLUSIONS

Analysis of a beam subjected to torsional moment is considered in this chapter. Uniform torsion (also called St.Venant's torsion) applied to the beam would cause a twist. Non-uniform torsion will cause both twisting and warping of the cross section. Simple methods of evaluating the torsional effects are outlined and discussed.

7.0 REFERENCES

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